FLOW OF A VISCOUS HEAT-CONDUCTING GAS AT HIGH SUPERSONIC SPEEDS ABOUT A CONE

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In this paper the flow of a viscous heat-conducting gas about a circular cone without angle of attack at high supersonic speeds is studied. The whole disturbed region of flow is divided into two sub-regions separated by a distinct boundary [1,2]: a viscous region in which the flow is considered to be laminar and is described by the boundary layer equations, and a non-viscous region in which the flow is described by the equations of an ideal gas.

Only the case of weak interaction is investigated, i.e. the region of flow considered is sufficiently far removed from the nose of the cone. For this the quantity $\epsilon = \delta/\beta L$ is considered to be small, where L is the distance of this region from the nose along the axis of a cone of half-opening angle θ_0, δ is the thickness of the boundary layer at this point and $\beta_0 = \tan\theta_0$, and the problem is solved by the method of small perturbations (in a construction similar to earlier solutions of the problem of a plate [3,4] and a wedge [5]). Terms of order ϵ^2 and higher are not taken into consideration.

The solution of the equations of an infinitely thin boundary layer on a cone – this problem reduces to the problem of a plate [6] – and the tabulated conical flow of an ideal gas [7] represent the fundamental solution. The surface of the cone is assumed to be isothermal or heat-insulated.

We shall employ two systems of coordinates with origin at the apex of the cone: the cylindrical coordinates x_1, r, ϕ for the non-viscous region and the conical coordinates x, y, ϕ for the viscous. Here x_1 is measured along the axis of the cone, x along the generator; r is the distance from a point to the axis, y to the surface of the cone. We shall designate by u, $u^{(1)}$ and v, $v^{(1)}$ the projections of the velocity on the axes x_1, x and r, y respectively. The indices ∞, δ, w and k will refer respectively to quantities of the oncoming flow, to the edge of the boundary layer, to the surface of the cone and to dimensional quantities at the surface of the cone in the case of flow of an ideal gas about it.

1. Non-viscous region. The flow about a circular cone is described in the variables $(x_1,\ \beta = r/x_1)$ by the system

$$v_0' = -\frac{a_0^2 v_0}{\beta\,[a_0^2(1+\beta^2) - (\beta u_0 - v_0)^2]} \qquad \left(a_0^2 = \varkappa\frac{p_0}{\rho_0}\right)$$
$$\frac{p_0'}{p_0} = \varkappa\frac{\rho_0'}{\rho_0} = \varkappa\frac{\beta u_0 - v_0}{a_0^2}v_0', \qquad u_0' = -\beta v_0' \tag{1.1}$$

Here primes indicate derivatives with respect to β; density, velocities, pressure and distances are referred respectively to ρ_∞, U_∞, $\rho_\infty U_\infty^2$ and L.

For $\beta = \beta_0 = \tan\theta_0$ we have

$$v_0 = \beta_0 u_0, \qquad v_0' = -\frac{u_0}{1+\beta_0^2}, \qquad p_0' = \rho_0' = 0$$
$$u_0^{(1)\prime} = u_0'\cos\theta_0 + v_0'\sin\theta_0 = 0 \tag{1.2}$$

In the neighborhood of $\beta \approx \beta_0$ for a thin cone integration of the first equation of (1.1) taking (1.2) into account gives $v_0 = u_0\beta_0^2/\beta$. This formula gives good agreement with the exact solution up to the compression shock (with an accuracy of about 5 per cent for $M_\infty\beta_0 \approx 1$ and exact agreement for larger values of $M_\infty\beta_0$). At the compression shock the quantities v_0, p_0, ρ_0 and u_0 are related to the angle θ^* of inclination of the shock to the axis of the cone through the usual relations.

To determine quantities of first order of smallness we consider the cone to be sufficiently thin and we employ the system of equations simplified on the bases of the law of plane sections [8].

$$-x_1\frac{\partial v}{\partial x_1} + (\beta - v)\frac{\partial v}{\partial\beta} = \frac{1}{\rho}\frac{\partial p}{\partial\beta}, \qquad x_1\beta\frac{\partial\rho}{\partial x_1} - \beta^2\frac{\partial\rho}{\partial\beta} + \frac{\partial(\rho\beta v)}{\partial\beta} = 0$$
$$x_1\left(\frac{1}{p}\frac{\partial p}{\partial x_1} - \frac{\varkappa}{\rho}\frac{\partial\rho}{\partial x_1}\right) + (v-\beta)\frac{\rho^\varkappa}{p}\frac{\partial}{\partial\beta}\left(\frac{p}{\rho^\varkappa}\right) = 0 \tag{1.3}$$

The value of u is determined from the Bernoulli equation. We seek a solution of (1.3) in the form

$$v = v_0 + \varepsilon\frac{v_1}{\sqrt{x_1}} + O(\varepsilon^2), \qquad \rho = \rho_0 + \varepsilon\frac{\rho_1}{\sqrt{x_1}} + O(\varepsilon^2)$$
$$u = 1 + \varepsilon\frac{u_1}{\sqrt{x_1}} + O(\varepsilon^2), \qquad p = p_0 + \varepsilon\frac{p_1}{\sqrt{x_1}} + O(\varepsilon^2)$$
$$T = T_0 + \varepsilon\frac{T_1}{\sqrt{x_1}} + O(\varepsilon^2)$$

The functions $v_1(\beta)$, $p_1(\beta)$ and $\rho_1(\beta)$ satisfy the system

$$(\beta - v_0)\, v_1' + \left(\frac{1}{2} - v_0'\right) v_1 = \frac{1}{\rho_0} p_1' - \frac{p_0'}{\rho_0}\frac{\rho_1}{\rho_0}$$
$$(\rho_0 \beta v_1 + v_0 \beta \rho_1)' - \beta\left(\frac{1}{2}\rho_1 + \beta\rho_1'\right) = 0 \qquad (1.4)$$
$$\frac{1}{2}\left(\frac{p_1}{p_0} - \varkappa\frac{\rho_1}{\rho_0}\right) + (\beta - v_0)\left(\frac{p_1}{p_0} - \varkappa\frac{\rho_1}{\rho_0}\right)' = 0$$

In these equations we take the solution of equations (1.1) for v_0, p_0, etc.

Correspondingly, the velocity field in the boundary layer can be represented in the form

$$u^{(1)} = u_0^{(1)} + \varepsilon u_1^{(1)} + \ldots, \qquad v^{(1)} = \varepsilon v_0^{(1)} + \ldots \qquad \left(u_0^{(1)} \to \frac{u_0(\beta_0)}{\cos\theta_0} \text{ at } y \to y_\delta\right)$$

From the condition of continuity of velocity at $y = y_\delta$ we have

$$\beta_0 u_0(\beta_0) + \varepsilon u_1^{(1)} \sin\theta_0 + \varepsilon v_0^{(1)}(y_\delta)\cos\theta_0 =$$
$$= v_0(\beta_0) + \frac{y_\delta}{x_1} v_0' + \varepsilon\frac{v_1(\beta_0)}{\sqrt{x_1}} + O(\varepsilon^2), \qquad u_1^{(1)} \sim \beta_0^2 \qquad (1.5)$$

As is well-known [6]

$$v_0^{(1)}(y_\delta) = \frac{u_0}{\cos\theta_0}\left(\sqrt{3}v_p - \frac{y_\delta}{x}\right) \qquad \left(v_p = \frac{V_0}{\sqrt{3x}}\right)$$

where v_p is the dimensionless normal component of the velocity at the edge of the boundary layer on a plate. Hence

$$\varepsilon v_1(\beta_0) = V_0[1 + O(\varepsilon + \beta)] \approx V_0 \qquad (1.6)$$

Let the equations of the form of the shock and of the angle of its inclination be

$$r = \beta^* x_1 + \varepsilon r_1 + \ldots, \qquad \theta = \theta^* + \frac{\varepsilon\theta_1}{\sqrt{x_1}} + \ldots$$
$$r_1 = 2\theta_1\sqrt{x_1}, \qquad \beta^* = \operatorname{tg}\theta^* \approx \theta^*$$

Then, after linearizing the relations at the shock, we obtain for $\beta = \beta^*$

$$p_1(\beta^*) = \left[\frac{4\theta^*}{\varkappa + 1} - 2p_0'(\beta^*)\right]\theta_1$$
$$v_1 = \left[\frac{2}{\varkappa + 1}\left(1 + \frac{1}{M_\infty^2\theta^{*2}}\right) - 2v_0'(\beta^*)\right]\theta_1 \qquad (1.7)$$
$$\rho_1 = \left[\frac{4\rho_0}{2 + (\varkappa - 1)M_\infty^2\theta^{*2}} - 2\theta^*\rho_0'(\beta^*)\right]\frac{\theta_1}{\theta^*}$$

Equations (1.4) do not change form with the substitution of p_1/θ_1, ρ_1/θ_1 and v_1/θ_1 for p_1, ρ_1 and v_1; in this sense θ_1 vanishes in (1.7) and the problem reduces to a Cauchy problem with given values at the point $\beta = \beta^*$. The quantity θ_1 is determined from (1.6).

The last equation of (1.4) has an integral, which for $v_0 = u_0\beta_0^2/\beta$ and taking (1.7) into account assumes the form:

$$\frac{p_1}{p_0} - \varkappa\frac{\rho_1}{\rho_0} = k\left(\frac{\beta^2}{\beta_0^2} - 1\right)^{-1/4} = \omega(\beta)$$
$$k = \frac{4\varkappa(\varkappa-1)(M_\infty^2\theta^{*2}-1)^2(\beta^{*2}/\beta_0^2-1)^{1/4}}{(2\varkappa M_\infty^2\theta^{*2}-\varkappa+1)[2+(\varkappa-1)M_\infty^2\theta^{*2}]}\frac{\theta_1}{\theta^*} \tag{1.8}$$

Taking (1.8) into account the system (1.4) reduces to a system of two equations:

$$\left(\frac{p_1}{p_0}\right)' = \frac{\varkappa v_0'\beta}{v_0 a_0^2}\left(\frac{1}{2} + 2v_0'\right)v_1 - \frac{\varkappa v_0'\beta}{v_0 a_0^2}(\beta - v_0) \times$$
$$\times\left(\frac{1}{2\varkappa} - \frac{\varkappa-1}{\varkappa}v_0'\right)\frac{p_1}{p_0} + \frac{v_0'^2\beta}{a_0^2 v_0}(\beta - v_0)\,\omega(\beta)$$
$$v_1' = \left[-\frac{1}{\beta} - \frac{\beta - v_0}{a_0^2}v_0' + \frac{v_0'\beta}{v_0 a_0^2}(\beta - v_0)\left(\frac{1}{2} + 2v_0'\right)\right]v_1 +$$
$$+\left[\frac{1}{2\varkappa} - \frac{v_0'\beta}{v_0 a_0^2}(\beta - v_0)^2\left(\frac{1}{2\varkappa} - \frac{\varkappa-1}{\varkappa}v_0'\right)\right]\frac{p_1}{p_0} + \frac{v_0'^2\beta}{\varkappa v_0 a_0^2}(\beta - v_0)^2\omega(\beta) \tag{1.9}$$

The temperature in the stream is determined from the equation of state and formula (1.8):

$$T_1 = T_{11} + T_{12}, \qquad \frac{T_{11}}{T_0} = \frac{\varkappa - 1}{\varkappa}\frac{p_1}{p_0}, \qquad \frac{T_{12}}{T_0} = \frac{1}{\varkappa}\,\omega(\beta) \tag{1.10}$$

Equations (1.9) do not have singular points, therefore p_1 and T_{11} are bounded; while, on the contrary, $T_{12} \to \infty$ as $\beta \to \beta_0$. The quantities $T_{11}/\sqrt{x_1}$ and $T_{12}/\sqrt{x_1}$ represent, respectively, potential and vortical (constant along stream-lines) elements.

The unbounded growth of T_{12} is evidence of the invalidity of applying the method of small perturbations for $\beta = \beta_0$. However, one can not be concerned with the behavior of the solution in a region which in reality is occupied by the boundary layer. For at the edge of the boundary layer the value of T_{12} will be of normal order because of the small negative power in the expression for ω. From the Bernoulli equation and (1.10) it follows that

$$u_1^{(1)} = u_1\cos\theta_0 + v_1\sin\theta_0 = u_{11}^{(1)} + u_{12}^{(1)}$$

where

$$u_{11}^{(1)} = \frac{1}{\varkappa M^2 \sqrt{x_1}} \frac{p_1(\beta_0)}{p_0(\beta_0)}, \qquad u_{12}^{(1)} = \frac{\omega(\beta_0)}{\varkappa(\varkappa-1) M^2 \sqrt{x_1}}, \qquad M^2 = \frac{u_k^2 \rho_k}{\varkappa p_k \cos^2\theta_0} \tag{1.11}$$

According to the similarity law for the flow about thin bodies [9] for $M_\infty\theta_0 \sim 1$ the quantities p_0/θ_0^2, v_0/θ_0 and ρ_0 depend only on the variable

$$t = (\beta - \beta_0)/(\beta^* - \beta_0)$$

and the parameter $M_\infty\theta_0$. Let

$$P = (p_1/p_0)(\theta_0/\theta_1), \qquad W = v_1/\theta_1$$

Then equations (1.9) take the form

$$\begin{aligned} P' &= A_1(t) W + B_1(t) P + C_1(t) \\ W' &= A_2(t) W + B_2(t) P + C_2(t) \end{aligned} \tag{1.12}$$

The coefficients $A(t)$, $B(t)$, $C(t)$ and the quantities $P(1)$, $W(1)$, $k_0 = k\theta_0/\theta_1$ and $t^* = \beta^*/\beta_0$ will depend only on the parameter $M_\infty\theta_0$.

Consequently, the solution of the system (1.12) also depends only on $M_\infty\theta_0$.

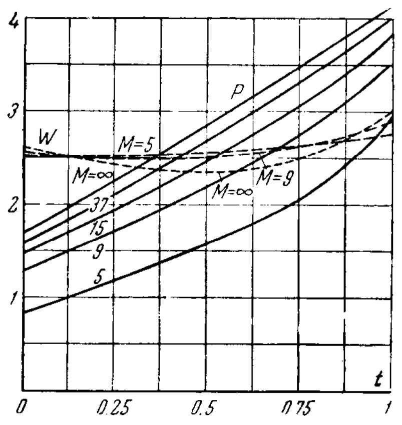

Fig. 1.

The functions $P(t)$ and $W(t)$ for various values of M are given in Fig. 1; the quantity t^* is given in Fig. 2. From (1.6) for $\beta = \beta_0$ it follows that

$$\varepsilon \frac{p_1}{p_0} = \frac{V_0}{\theta_0} \frac{P}{W}, \qquad \varepsilon\theta_1 = \frac{V_0}{W} \tag{1.13}$$

2. Viscous region: basic equations. The system of equations of a boundary layer of non-zero thickness on a cone has the form [2]

$$\rho u^{(1)} \frac{\partial u^{(1)}}{\partial x} + \rho v^{(1)} \frac{\partial u^{(1)}}{\partial y} = -\frac{dp}{dx} + \frac{1}{r} \frac{\partial}{\partial y}\left(r\mu \frac{\partial u^{(1)}}{\partial y}\right)$$

$$\rho u^{(1)} \frac{\partial i}{\partial x} + \rho v^{(1)} \frac{\partial i}{\partial y} = u^{(1)} \frac{dp}{dx} + \frac{1}{r} \frac{\partial}{\partial y}\left(r \frac{\mu}{\sigma} \frac{\partial i}{\partial y}\right) + \mu \left(\frac{\partial u^{(1)}}{\partial y}\right)^2$$

$$\frac{\partial}{\partial x}(\rho u^{(1)} r) + \frac{\partial}{\partial y}(\rho v^{(1)} r) = 0, \qquad r = x \cos\theta_0 + y \sin\theta_0 \tag{2.1}$$

Here i, μ and σ = const are the enthalpy, viscosity and Prandtl number*, respectively. Modifying the well-known Crocco transformation, we shall replace the variables (x, y) by $(\xi = x^3/3,\ u^{(1)})$ and the unknown function $u^{(1)}$ by $\tau = (\mu/x)(\partial u/\partial y)$. Introducing the stream function Ψ, we obtain from the first and last equations of (2.1)

$$\frac{1}{\beta_0}\frac{\partial \Psi}{\partial \xi} = \frac{\mu}{\tau}(1+K)\frac{dp}{d\xi} - \frac{\partial}{\partial u^{(1)}}[\tau(1+K)], \quad \frac{1}{\beta_0}\frac{\partial \Psi}{\partial u^{(1)}} = \frac{\rho\mu u^{(1)}}{\tau}(1+K) \tag{2.2}$$

Here $K = y/\beta_0 x \sim \epsilon$. In what follows we shall everywhere replace the dimensional quantities $u^{(1)}$, ρ, i, μ, p, τ, x, y without changing their designations by the corresponding dimensionless quantities:

$$\frac{u^{(1)}\cos\theta_0}{u_k}, \quad \frac{\rho}{\rho_k}, \quad \frac{i}{i_k}, \quad \frac{\mu}{\mu_k}, \quad \frac{p}{\rho_\infty U_\infty^2}, \quad \frac{\tau \cos^2\theta_0\ L\sqrt{R}}{\rho_k u_k^2}, \quad \frac{x}{L}, \quad \frac{y}{L}, \quad R = \frac{\rho_k u_k L}{\mu_k \cos\theta_0}$$

We shall make the transformation of variables $(\xi,\ u^{(1)}) \to (\xi,\ \eta = u^{(1)}/u_\partial^{(1)})$ and we shall introduce the designations

$$F(i,\ p) = \frac{\mu\rho}{\mu_k,\ \rho_k}, \qquad \frac{\partial F(i,\ p)}{\partial p} = f, \qquad K = \varepsilon K_0, \qquad u_\delta^{(1)} = u_\delta$$

(For air over a wide range of temperatures $\partial F/\partial i \ll 1$, and this will be exploited below.) Then after eliminating Ψ from (2.2) and after the usual transformation we obtain the system

$$\frac{\partial^2}{\partial \eta^2}[(1+\varepsilon K_0)\tau] + \eta u_\delta^3 \frac{\partial}{\partial \xi}\left(\frac{1+\varepsilon K_0}{\tau} F\right) = \tag{2.3}$$

$$= \frac{u_\delta}{\varkappa M^2}\frac{\partial}{\partial \eta}\left[\frac{\mu(1+\varepsilon K_0)}{\tau}\right]\frac{d}{d\xi}\frac{p}{p_0} + u_\delta^2 \eta^2 \frac{du_\delta}{d\xi}\frac{\partial}{\partial \eta}\left(F\frac{1+\varepsilon K_0}{\tau}\right)$$

$$\tau^2 \frac{\partial^2 i}{\partial \eta^2} + (1-\sigma)\tau \frac{\partial \tau}{\partial \eta}\frac{\partial i}{\partial \eta} + (\varkappa - 1) M^2 \sigma u_\delta^2 \tau^2 - \sigma \eta u_\delta^3 F \frac{\partial i}{\partial \xi} =$$

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* For gases $\sigma = \sigma(i,\ p)$ is a slowly varying function and the influence of the variation of σ on the solution can be studied by the method of [10].

$$= -\frac{\sigma}{\varkappa M^2}\mu u_\delta\left[(\varkappa-1)M^2\eta u_\delta^2+\frac{\partial i}{\partial\eta}\right]\frac{d}{d\xi}\frac{p}{p_0}-$$

$$-\sigma\eta^2 u_\delta^2 F\frac{du_\delta}{d\xi}\frac{\partial i}{\partial\eta}-\varepsilon(1-\sigma)\frac{\tau^2}{1+\varepsilon K_0}\frac{\partial K_0}{\partial\eta}\frac{\partial i}{\partial\eta}$$

The first equation of (2.1) and the temperature conditions give for $\eta = 0$

$$\tau\frac{\partial}{\partial\eta}\left[\tau(1+\varepsilon K_0)\right]=\frac{u_\delta\mu}{\varkappa M^2}(1+\varepsilon K_0)\frac{d}{d\xi}\frac{p}{p_0},\qquad i=i_w \quad \text{или} \quad \frac{\partial i}{\partial\eta}=0 \quad (2.4)$$

According to (1.2) one should set

$$p/p_0 = 1+\varepsilon p_1(\beta_0)/\sqrt{x}\,p_0(\beta_0)$$

in (2.3) and (2.4).

3. Viscous region: solution of the equations. We shall seek a solution of the system (2.3) in the form

$$\tau=\tau_*+\varepsilon\tau_1+\ldots,\qquad i=i_*+\varepsilon i_1+\ldots \quad (3.1)$$

Let $\tau_* \to 0$, $i_* \to 1$ as $\eta \to 1$. Then, setting

$$\tau_*=g_*(\eta)\sqrt{2\xi},\qquad i_*=i_*(\eta)$$

and rejecting quantities of order ϵ in (2.3), we have for $g_*(\eta)$ and $i_*(\eta)$

$$g_*g_*''=-\eta F(i_*,\ p_0),\quad (g_*'(0)=g_*(1)=0),\quad i_*=1+\sigma(\varkappa-1)M^2J_1+eJ_2$$

$$J_1=\int_\eta^1 g_*^{\sigma-1}\int_0^\eta g_*^{1-\sigma}d\eta\, d\eta,\qquad J_2=\int_\eta^1 g_*^{\sigma-1}d\eta,\qquad e=\frac{i_w-i_e}{J_2(0)}$$

$$i_e=1+\sigma(\varkappa-1)M^2J_1(0) \quad (3.2)$$

The solution of this system for an arbitrary form of F is considered in references 10 and 11.

We shall consider the conditions for τ_1 and i_1 at the outer edge. Transition from the $\xi\eta$-plane to the physical plane, i.e. to the xy-plane, is effected for $\epsilon \neq 0$ and for $\epsilon = 0$, respectively, by the formulas

$$y=\frac{u_\delta}{x\sqrt{R}}\int_0^\eta\frac{\mu}{\tau}d\eta,\qquad y=\sqrt{\frac{2x}{3R}}\int_0^\eta\frac{\mu(i_*,\ p_0)}{g_*}d\eta=\sqrt{\frac{2x}{3R}}\,h \quad (3.3)$$

To some fixed point $(x,\ y)$ there corresponds a value of $\eta(x,\ y)$

$$\eta(x,\ y)=\eta_0(y/\sqrt{x})+\varepsilon\eta_1(x,\ y)=\frac{u_0^{(1)}+\varepsilon u_1^{(1)}}{1+\varepsilon u_{1\delta}}=$$
$$=u_0^{(1)}+\varepsilon(u_1^{(1)}-u_0^{(1)}u_{1\delta}) \tag{3.4}$$

The function $\eta_0(y/\sqrt{x})$ is determined from the second formula of (3.3), and η_1 from the condition of equality of the right-hand sides of both formulas. At this same point the functions τ and i are equal to $\tau_*(\eta_0)$ and $i_*(\eta_0)$ for $\epsilon = 0$ and to

$$\tau=\tau_*(\eta_0)+\varepsilon\left(\eta_1\frac{\partial\tau_*}{\partial\eta}+\tau_1\right)\ldots,\qquad i=i_*(\eta_0)+\varepsilon(i_1+i_*'\eta_1) \tag{3.5}$$

in the general case. Near the edge we have $\partial u_0^{(1)}/\partial\beta\sim\epsilon\beta$, $\partial T_0/\partial\beta\sim\epsilon\beta$ for the behavior from within [see (1.2] and

$$\partial u_0^{(1)}/\partial y\sim g_*\approx 0,\qquad \partial i_*/\partial y\sim g^\sigma{}_*\approx 0$$

for the behavior from without. Consequently, the boundary conditions do not change upon transferring them from the true edge of the boundary layer to the y_δ-edge at $\epsilon = 0$, determined according to (3.3), and

$$u_{1\delta}=u_{11}^{(1)}+u_{12}^{(1)},\qquad i=i_*+\varepsilon T_1/T_0\sqrt{x}\quad \text{at}\quad y=y_\delta$$

Because [10]

$$u_0^{(1)}=1+g_*/g_*'\approx 1\quad \text{at}\quad \eta\approx 1$$

the boundary conditions for τ_1 and i_1 in accordance with (3.4) and (3.5) must conform to the requirement

$$\eta_1\approx 0,\qquad i_1+\eta_1 i_*'\approx T_1/T_0\sqrt{x}\quad \text{at}\quad \eta=\eta_\delta\approx 1 \tag{3.6}$$

moreover, we will require (and, consequently, determine y_δ) that these conditions be satisfied at $\eta_\delta = 0.98 \div 0.999$.

Substituting (3.1) into (2.3) and setting the sum of the terms of order ϵ equal to zero, we obtain a system of linear partial differential equations for τ_1 and i_1, which can be separated into three independent groups setting

$$\tau_1=\tau_{11}+\tau_{12}+\tau_{13},\quad i_1=i_{11}+i_{12}+i_{13},\quad \eta_1=\eta_{11}+\eta_{12}+\eta_{13}$$

Here τ_{11}, i_{11} and τ_{12}, i_{12} are stipulated, respectively, by the potential and vortical components of the non-viscous flow, and τ_{13}, i_{13} by the presence of the quantity K_0 in the equations. In accordance with (1.10) and (3.3) we have for $\beta = \beta_\delta$

$$\frac{T_{12}}{T_0} = \frac{k}{\varkappa}\,\omega(\eta_\delta) = \frac{k}{\varkappa}\left(\frac{2y_\delta}{x\beta_0}\right)^{-1/4} = \varepsilon^{-1/4}\frac{k}{\varkappa}\left(\frac{2}{\beta_0}\sqrt{\frac{2}{3R}}\right)^{-1/4} h_\delta^{-1/4}x^{1/8}, \qquad h_\delta = h(\eta_\delta)$$

Because $y_\delta/\beta_0 x \sim \epsilon$, the quantity $\epsilon T_{12}/T_0\sqrt{x}$ is formally of the order of smallness $\epsilon^{3/4}$ at $y = y_\delta$, i.e. its order is lower than that of $\epsilon T_{11}/T_0\sqrt{x}$. However, this difference is immaterial, since $\epsilon^{-1/4} \sim 1$ for values of ϵ of practical interest. Noting that

$$x^{1/2} = 3^{1/6}\xi^{1/6}, \qquad \varepsilon K_0 = \frac{y}{\beta_0 x} = \frac{h}{\beta_0}\left(\frac{3}{2}Rx\right)^{-1/2}$$

we will seek solutions in the form

$$\tau_{11} = \frac{3^{-1/6}\xi^{-1/6}p_1}{\varkappa M^2\sqrt{2\xi}\,p_0}\,g_1(\eta), \qquad i_{11} = \frac{3^{-1/6}\xi^{-1/6}p_1}{\varkappa M^2 p_0}\left[j_1(\eta) + (\varkappa-1)M^2(1-2\sigma J_1)\right]$$

$$\tau_{12} = \frac{3^{1/6}k\xi^{-1/6}}{\varkappa(\varkappa-1)M^2\sqrt{2\xi}}\left(\frac{\beta_0\sqrt{R}}{2\sqrt{2}}\right)^{1/4} g_2(\eta), \qquad \varepsilon\tau_{13} = \frac{3^{-2/3}\xi^{-1/6}}{\beta_0\sqrt{R\xi}}\,g_3(\eta)$$

$$i_{12} = \frac{3^{-1/6}k\xi^{-1/6}}{\varkappa}\left(\frac{\beta_0\sqrt{R}}{2\sqrt{2}}\right)^{1/4}\left[j_2(\eta) - 2\sigma h_\delta^{-1/4}J_1\right] \tag{3.7}$$

$$\varepsilon i_{13} = \frac{3^{-2/3}\sqrt{2}\xi^{-1/6}}{\beta_0\sqrt{R}}\,j_3(\eta), \qquad \eta_{11} = \frac{3^{-1/6}\xi^{-1/6}p_1}{\varkappa M^2 p_0}\,\eta_{*1}(\eta)$$

$$\eta_{12} = \frac{3^{-1/6}k\xi^{-1/6}}{\varkappa}\left(\frac{\beta_0\sqrt{R}}{2\sqrt{2}}\right)^{1/4}\eta_{*2}(\eta), \qquad \varepsilon\eta_{13} = \frac{3^{-1/6}\xi^{-1/6}\sqrt{2}}{\beta_0\sqrt{3R}}\,\eta_{*3}(\eta)$$

Thus, the problem reduces to the solution of three systems of the form:

$${}^2g_m'' - \alpha_m\eta F g_m = \varphi_m, \qquad g_m{}^2 j_m'' + (1-\sigma)g_* g_*' j_m' + \beta_m\sigma\eta F j_m = \psi_m \tag{3.8}$$

$$\left(m = 1,\ 2,\ 3;\ \alpha_1 = \alpha_3 = \frac{2}{3},\ \ \alpha_2 = \frac{3}{4},\ \ \beta_1 = \beta_3 = \frac{1}{3},\ \beta_2 = \frac{1}{4}\right)$$

$$\varphi_1 = \left(3F - \frac{2}{3}\varkappa/M^2\right)\eta g_* - \frac{1}{3}g_*{}^2\left[\left(\frac{\mu}{g_*}\right)' - \eta^2\left(\frac{F}{g_*}\right)'\right]$$

$$\varphi_2 = h_\delta^{-1/4}F\left(3\eta g_* - \frac{1}{4}\eta^2 g_*'\right), \qquad \varphi_3 = -g_*{}^2(hg_*)'' - \frac{2}{3}\eta g_* F h$$

$$\psi_1 = -(1-\sigma)\,i_*' g_*{}^2\left(\frac{g_1}{g_*}\right)' + \frac{2}{3}\sigma^2(\varkappa-1)M^2\eta F J_1 +$$

$$+ \frac{1}{3}\sigma(\varkappa-1)M^2(\mu - F)\eta + \frac{1}{3}\sigma(\mu - \eta^2 F)\,i_*'$$

$$\psi_2 = -\frac{1-\sigma}{(\varkappa-1)M^2}g_*{}^2 i_*'\left(\frac{g_2}{g_*}\right)' + \frac{1}{2}\sigma^2\eta F h_\delta^{-1/4} - \frac{\sigma}{4(\varkappa-1)M^2}\eta^2 F i_*' h_\delta^{1/4}$$

$$\psi_3 = -(1-\sigma)g_*{}^2 i_*'\left(h + \frac{g_3}{g_*}\right)'$$

Here and in what follows*

$$F = F(i_*,\ p_k), \qquad f = f(i_*,\ p_k), \qquad \mu = \mu(i_*,\ p_k)$$

From (3.7) and (3.3) it follows that

$$\eta_{*m} = \frac{g_*}{\mu}\left(\int_0^\eta \frac{\mu g_m}{g_*^2}\,d\eta - \int_0^\eta [j_m - (\varkappa - 1) M^2 (1 - \sigma J_1)] \frac{\mu'}{g_*}\,d\eta - \int_0^\eta \frac{\mu}{g_*}\,d\eta\right) \qquad (3.9)$$

Taking (3.7) into account the conditions (3.6) and (2.4) take the form:

$$j_1 - 2\sigma(\varkappa - 1) M^2 J_1 + \eta_{*1} i_*' = 0, \qquad j_2 - 2\sigma h_\delta^{-1/4} J_1 + \eta_{*2} i_*' = h_\delta^{-1/4}$$

$$j_3 + \eta_{*3} i_*' = 0, \qquad \eta_{*m} = 0 \qquad \text{при} \quad \eta = \eta_\delta \qquad (3.10)$$

$$g_* g_1' = -\tfrac{1}{3}\mu, \qquad g_2' = 0, \qquad g_3' = -\mu \qquad (3.11)$$

$$j_1 + (\varkappa - 1) M^2 = 2\sigma(\varkappa - 1) M^2 J_1, \qquad j_2 = 2\sigma h_\delta^{-1/4} J_1,$$

$$j_3 = 0 \quad \text{или} \quad j_m' = 0 \quad \text{при } \eta = 0$$

Let g_m^* and j_m^* be solutions of equations (3.8) for $\phi_m = \psi_m = 0$, which satisfy the conditions $g_m^*(0) = j_m^*(0) = 1$ and $g_m^{*\prime}(0) = j_m^{*\prime}(0) = 0$. Then the general solution of equations (3.8) has the form (A_m, B_m and C_m are constants):

$$g_m = -g_m^* \int_\eta^1 \frac{1}{g_m^{*2}} \int_0^\eta \frac{\varphi_m g_m^*}{g_*^2}\,d\eta\,d\eta - g_m'(0)\,g_m^{**} + A_m g_m^* \qquad (3.12)$$

$$j = j_m^* H_m(\eta) + C_m j_m^{**} + B_m j_m^* \qquad (3.13)$$

$$g_m^{**} = g_m^* \int_\eta^1 \frac{d\eta}{g_m^{*2}}, \qquad j_m^{**} = j_m^* \int_\eta^1 \frac{g_*^{\sigma-1}}{j_m^{*2}}\,d\eta, \qquad H_m(\eta) = \int_0^\eta \frac{g_*^{\sigma-1}}{j_m^{*2}} \int_0^\eta \frac{\psi_m j_m^*}{g_*^{\sigma+1}}\,d\eta\,d\eta$$

Here g_m^{**} and j_m^{**} are solutions, which are linearly independent of g_m^* and j_m^*, of the same homogeneous equations. It is easy to verify that as $\eta \to 1$ the solutions of the first and second homogeneous equations of (3.8) have the form

$$\text{const}\,(-g_*')^{\alpha_m}, \qquad \text{const}\, g_*(-g_*')^{-(1+\alpha_m)}$$

$$\text{const}\,(-g_*')^{-\beta_m}, \qquad \text{const}\, g_*^{\sigma}(-g_*')^{\beta_m - 1}$$

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* Without the restriction that $\partial F/\partial i \ll 1$ the right-hand side of ϕ_m would contain terms of the form $f_m(\partial F/\partial i)$ and the solution of the system (3.8) would be considerably complicated.

From the structure of the equation it follows that g_m^* is an increasing function and, consequently, has the form: const $(-g_*')^{\alpha_m}$ for $\eta \approx 1$. Both of the solutions j_m^* and j_m^{**} decrease as $\eta \to 0$, but as shown below in the appendix $j_m^* \to$ const $(-g_*')^{-\beta_m}$ exactly. For $\eta \approx 1$ and $\sigma < 2$ the following is valid*:

$$\mu = 1 + \mu'(i_* - 1), \quad \varphi_1 = \text{const}\,(1-\sigma)\,g_*^{\sigma} + \text{const}\,g_*$$

$$i_* - 1 = \text{const}\left(\frac{g_*^{\sigma}}{g_*'}\right), \qquad g_1 \approx A_1 g_1^* + \text{const}\left(\frac{g_*^{\sigma}}{g_*'^2}\right) + \text{const}\ g_*$$

$$\psi_1 \approx \text{const}\ A_1 g_*^{\sigma-1}(-g_*')^{5/3} + \text{const}\,(-g^{2\sigma-1}/g_*') + \text{const}\,(-g_*^{\sigma}/g_*')$$

The integral H_1 converges for $\sigma > 1/2$ and diverges for $\sigma < 1/2$. Putting $B_1 = B_{12} + B_{11}$, where $B \quad = -H(1)$ for $\sigma > 1/2$ and $B_{11} = 0$ for $\sigma < 1/2$, we have for $\eta \approx 1$

$$j_1 = \text{const}\ A_1 j_1^* g_*^{\sigma-1} + \text{const}\ g_*^{2\sigma-1}\,[g_*'^3(\sigma-1)(2\sigma-1)]^{-1} + \\ + \text{const}\ g_*^{\sigma}(-g_*')^{-1} + B_{12} j_1^* \tag{3.14}$$

For $\sigma < 1$ these formulas are valid for those values of η for which $(1-\sigma)g_*'^2 \gtrsim 1$, $(2\sigma - 1)g_*'^2 \gtrsim 1$. We obtain the asymptotic form for j_{*1} at $\sigma \approx 1$ or $\sigma \approx 1/2$ by formally setting $(1-\sigma)g_*'^2 = 1$ or $(2\sigma - 1)g_*'^2 = 1$, respectively, in (3.14). From (3.9) one can obtain

$$\eta_{*1} = \text{const}\ A_1(-g_*')^{-1/3} + \text{const}\,g_*^{\sigma}\,[g_*'^3(\sigma-1)]^{-1} + \text{const}\,g_* g_*'.$$

The function $(-g_*')^{-1/3}$ decreases very slowly ($(-g_*')^{-1/3} = 0.69$ for $\eta = 0.999$ if $F = 1$). Therefore, from the condition that $\eta_{1*} \approx 0$ it follows that $A_1 = 0$. For $\eta \approx 1$ a reasonable relation, which we advance without proof, is

$$j_m^* H_m + \eta_{*m} i_*' = \text{const}\ j_m^* + O(g_*^{\sigma} g_*') \tag{3.15}$$

This relation gives the possibility of determining B_m from the conditions (3.10) for any value of σ. If $\sigma > 1/2$, then $\eta_{*1} i_*' \to 0$ and $j_1 \approx 0$ as $\eta \to 1$; consequently** $B_1 = -H_1(1)$.

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* The correctness of the asymptotic evaluations cited below can be demonstrated by successive application of the L'Hopital rule and the Cauchy mean value theorem.

** For $m = 1$ the equations (3.8) are identical with the equations of [10], where $\alpha_1 = 2M + 1$, $\beta_1 = -2M$, and M is real. In [10] the condition $j_{m1} = 0$ for $\eta = 1$ is used. This condition is valid as stated only for $\sigma > 1/2$, and for greater rigor (3.10) would be used.

For $m = 2$ the solution of equation (3.8) has the form:

$$g_2 = -h_\delta^{-1/4}\eta g_*' + A_2 g_2^*$$

which when taken into account gives for $\eta \approx 1$

$$\eta_{*2} \approx h_\delta^{-1/4} + \frac{4}{3} A_2 g_* g_2^{*\prime}, \quad g_2^* \approx a_0 (-g_*')^{3/4}, \quad (h_\delta = -g_*'(\eta_\delta) + h_0, \ h_0 = \text{const})$$

Setting $\eta_{*2}(\eta_\delta) = 0$ and consequently $j_2(\eta_\delta) \approx h_\delta^{-1/4}$, we will have

$$A_2 = -\frac{3}{4}(h^{1/4} g_* g_2^{*\prime})^{-1}_{\eta=\eta_\delta} \approx -\frac{1}{a_0}\left[\frac{-g_*'(\eta_\delta)}{h_\delta}\right]^{1/4}, \quad B_2 = \frac{h_\delta^{-1/4}}{j_2^*(\eta_\delta)} - H_2(\eta_\delta) \quad (3.16)$$

Within the limits $\eta_\delta = 0.98 - 0.999$ the value of A_2 does not change materially; however, because $h_0 \gg -g_*'(\eta_\delta)$ for $M \gg 1$, it is important to distinguish it from its limiting value as $\eta_\delta \to 1$ which is equal to $-1/a_0$. Although H_2 has the singularity of form $[g^{\sigma-1}/(1-\sigma)g_*'^2]$ as $\eta \to 1$, the value of B_2 also varies little for $\eta_\delta = 0.98 - 0.999$. According to (3.15) the value of B_2 is, as a matter of fact, finite at $\eta_\delta \to 1$, which is different from the value chosen by us in (3.16).

For $m = 3$ formula (3.12) transforms to

$$g_3 = -g_* h + H_0(\eta) + A_3 g_1^*$$

where $H_0(\eta)$ designates the first integral of the right-hand side of (3.12), in which $-4/3\,\phi\, hFg$ is substituted for the function ϕ_m. For $\eta \approx 1$

$$g_3 = A_3 g_1^* + O(g_* g_*'), \qquad j_3 = \text{const}\, A_3 j_1^* g_*^{\sigma-1} + H_3(1)\, j_1^* + O(g_*^\sigma / g_*'^2) + B_3 j_1^*$$

$$\eta_{*3} = \text{const}\, A_3 (-g_*')^{-1/3} + O(g_* g_*')$$

From (3.10) it follows that $A_3 = 0$ and $B_3 = -H_3(1)$. We note that $j_3 \equiv 0$ for $\sigma = 1$. The constants C_m are chosen from the conditions at $\eta = 0$ and are equal to zero for $j_m'(0) = 0$.

It should be noted that for $m = 1$ or $m = 3$, just as in the fundamental approximation, the requirement of continuity of the velocity and temperature fields at the transition from the viscous to the non-viscous region reduces essentially to the boundary conditions of the asymptotic boundary layer. In the case of $m = 2$, the scheme of a boundary layer asymptotic in the classical sense does not provide a practical coincidence of the parameters of the viscous and non-viscous solution at the rationally chosen limit of the division between them, because the trend of the functions to their maximum values at $\eta = 1$ is so slow ($g_2 \sim -(g_*')^{-1/4}$, $\eta_{*2} \sim (-g_*')^{-5/4}$ as $\eta \to 1$ if $A_2 = -1/a_0$) that these values can formally be attained at values of y far removed from the edge of the boundary layer.

In conclusion we will derive a formula for V_0. From the first and last equations of (1.1) for a plate (dp/dx)$= 0$, $r \to \infty$) there follows in dimensional quantities

$$\rho v = -\frac{\partial\Psi}{\partial x}\bigg|_u - \frac{\partial u}{\partial x}\bigg|_y \frac{\partial\Psi}{\partial u}\bigg|_x, \quad \frac{\partial\Psi}{\partial x}\bigg|_u = -\frac{\partial\tau_*}{\partial u}, \quad \frac{\partial\Psi}{\partial u}\bigg|_x = \frac{\rho\mu u}{\tau_*}, \quad \tau_* = \mu\frac{\partial u}{\partial y}$$

Substituting here $\partial u/\partial x = -(\partial y/\partial x)/(\partial y/\partial u)$ and transforming to dimensionless quantities we obtain

$$\rho v = \frac{g_*' + \rho\eta h}{\sqrt{2xR}}, \quad v_p = \frac{g_*' + h}{\sqrt{2xR}} = \frac{h_0}{\sqrt{2xR}}, \quad h_0 = \int_0^1 \frac{\mu - \eta F}{g_*}\, d\eta, \quad V_0 = h_0\sqrt{\frac{3}{2R}}$$

4. Analysis of results. To obtain more general, though also less exact, formulas for numerical calculations it was assumed that $F = F_0 =$ const, $\mu = F_0 i$. In this case [10]

$$\begin{aligned}
&g_* = \sqrt{F_0}g_0,\ h_0 = \sqrt{F_0}[I_0 + \sigma(\varkappa - 1)M^2 I_1 + eI_2],\ h = -\sqrt{F_0}g_0' + h_0 = \sqrt{F_0}h_1\\
&g_1 = \sqrt{F_0}[(0.4\varkappa M^2 - 1.8)g_0 + g_{11} + (\varkappa - 1)M^2 g_{12} + eg_{13}].\\
&g_2 = -F_0^{3/4} h_{1\delta}^{-1/4}\left[\eta g_0' + \frac{1}{a_0}(-g_0')_\delta^{1/4} g_2^*\right],\ g_3 = F_0[g_{31} + (\varkappa - 1)M^2 g_{32} + eg_{33}]\\
&j_1 = (\varkappa - 1)M^2 j_{11} + (\varkappa - 1)^2 M^4 j_{12} + (\varkappa - 1)M^2 e j_{13} + e^2 j_{14} + e j_{15} + C_1 j_1^{**} \quad (4.1)\\
&j_2 = (F_0 h_\delta)^{-1/4}\left[j_{21} + \frac{e}{(\varkappa - 1)M^2} j_{22} + C_2 j_2^{**}\right]\\
&j_3 = \sqrt{F_0}[(\varkappa - 1)M^2 j_{31} + (\varkappa - 1)^2 M^2 j_{32} + e(\varkappa - 1)M^2 j_{33} + e^2 j_{34} + e j_{35} + C_3 j_1^{**}]
\end{aligned}$$

The function $g_0(\eta)$ satisfies equation (3.2) for $F = 1$. All of the functions entering in the right-hand sides of (4.1) depend only on σ. For $\sigma = 0.725$ the values of some of these functions for $\eta = 0$ are listed in Table 1.

TABLE 1.

m	g_{m1}	g_{m2}	g_{m3}	j_{m1}	j_{m2}	j_{m3}	j_{m4}	j_{m5}	j_m^{**}
1	0.668	0.211	0.500	−0.0036	−0.0071	0.630	1.62	1.12	1.74
2				−0.543	0.97				1.67
3	0.488	0.174	0.450	−0.060	−0.110	−0.048	−0.061	−0.121	

$[I_0(1) = 1.22, \quad I_1(1) = 1.08, \quad I_2(1) = 1.87, \quad a_0 = 1,43]$

The magnitude of the induced pressure on the surface of the cone $(p - p_0)p_0$ and the thickness of the boundary layer are determined for $\sigma = 0.725$ and $\kappa = 1.4$ from the formula

$$\frac{p - p_0}{p_0} = \left(0.515 + 8.3\frac{i_w}{M_\infty^2 i_\infty} - 0.86\,\beta_0^2\alpha_2\right)\alpha_1\Omega \quad (4.2)$$

$$\frac{y_\delta}{\beta_0 x} = \left(0.34 + 5.5\frac{i_w}{M_\infty^2 i_\infty} + 10.7\,\beta_0^2\alpha_2\right)\alpha_3\Omega \quad (4.3)$$

In the equalities (4.2) and (4.3) the following designations have been assumed:

$$\Omega = \frac{\chi}{M_\infty{}^2 \beta_0{}^2} = \frac{\varkappa - 1}{2\beta_0{}^2} \sqrt{\frac{U_\infty \mu_\infty F_0 F_1}{\varkappa x p_\infty}}$$

$$\chi = \frac{\varkappa - 1}{2} M_\infty{}^3 \left(\frac{F_0 F_1}{R_\infty}\right)^{1/2}, \qquad F_1 = \frac{\mu_k T_\infty}{\mu_\infty T_k}, \qquad R_\infty = \frac{U_\infty \rho_\infty x}{\mu_\infty}$$

Here and in what follows i_w and x are dimensional quantities, the coefficients a_i depend only on $M_\infty \beta_0$ and are presented in Fig. 2. The heat flow to the surface in dimensional quantities is equal to $q = -xu_\delta (\tau/\sigma)(\partial i/\partial y)|_{\eta=0}$, from which there follows

$$\varepsilon \frac{q_1}{q_0} = \left(\frac{\varepsilon}{i_0}\frac{\partial i_1}{\partial \eta} + \varepsilon \frac{\tau_1}{\tau_0} - \varepsilon u_{1\delta}\right)_{\eta=0}, \quad q_1 = q_{11} + q_{12} + q_{13}$$

The quantities g_{11}, τ_{11}, etc. are determined for the same σ and κ from the formula

$$\begin{aligned}
\frac{\varepsilon\tau_{11}}{\tau_0} &= \left[4.55\left(\frac{i_w}{M_\infty{}^2 i_\infty}\right)^2 + 3.86\frac{i_w}{M_\infty{}^2 i_\infty} + 0.222\right]\alpha_1\Omega \\
\frac{\varepsilon\tau_{12}}{\tau_0} &= -\left(0.085 + 1.35\frac{i_w}{i_\infty M_\infty{}^2}\right)^{3/4} \alpha_4 \beta_0{}^2 \chi^{3/4} \\
\frac{\varepsilon\tau_{13}}{\tau_0} &= \left(2.84\frac{i_w}{i_\infty M_\infty{}^2} + 0.123\right)\alpha_3\Omega \\
\frac{\varepsilon q_{11}}{q_0} &= \left[-0.95\left(\frac{i_w}{i_\infty{}^2 M^2}\right)^2 + 3.27\frac{i_w}{i_\infty M^2} + 0.207\right]\alpha_1\Omega \\
\frac{\varepsilon q_{13}}{q_0} &= \left(2.98\frac{i_w}{i_\infty M_\infty{}^2} + 0.161\right)\alpha_3\Omega
\end{aligned} \tag{4.4}$$

The terms in (4.2) - (4.4) which contain $\beta_0{}^2$ are small and for $M_\infty \gg 1$ immaterial (in the formulas (4.4) these terms are omitted). This fact is confirmation of the similarity law of the supersonic flow of a viscous gas [2], in accordance with which similarity criteria there are the parameters

$$\chi, \; M_\infty \beta_0 \quad (i_w / i_\infty M_\infty{}^2)$$

The calculations showed that the equilibrium temperature of the surface is practically independent of ϵ and remains equal to its value at $\epsilon = 0$. The terms containing j_{11}, j_{12}, j_{31}, j_{32} are negligibly small in comparison to the others in the expressions for q_{1m} and are omitted in (4.4). From (4.4) it follows that $\tau_{12}/\tau_{11} \sim q_{12}/q_{11} \sim \beta_0{}^4 M_\infty^{5/4} R_\infty^{1/8} \ll 1$ for $R_\infty < 10^8$ and $M_\infty < 20$. Consequently, the turbulence of the flow due to the curvature of the shock wave does not show an appreciable effect on the characteristics of the boundary layer.

From the formula (4.4) it follows than an increase in the boundary layer thickness for $M_\infty \gg 1$ leads to an increase in the frictional resistance and, particularly important, to an increase in the heat flow to the surface of the body.

It should be particularly noted that τ_{13}, q_{13} have the same order of magnitude as τ_{11}, q_{11}. This means that on axi-symmetric bodies, in contrast to plates, the thickening of the boundary layer for $M_\infty \gg 1$ by itself irrespective of the rise in pressure due to the interaction leads to an increase in the frictional resistance and the heat transfer. This phenomenon can be shown to be important also in those cases in which the effect of the boundary layer on the external flow will not play an important part, for instance, on the forward part of the lateral surface of a blunt body.

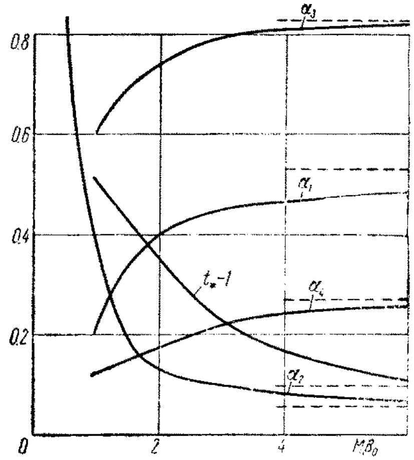

Fig. 2.

The limit of application of the obtained results can be estimated from the condition $y_\delta/\beta_0 x \ll 1$. In order that one may neglect a quantity of order ϵ^2 it is sufficient, for instance, that $\epsilon = y_\delta/\beta_0 x < 1/7$.

In this case for $F_0 = F_1 = 1$ it must follow from (4.3) that

$$R_\infty \geqslant 0.15\, M_\infty^2 / \beta_0^4 \text{ or } x \gg 0.1\,(U_\infty \mu_\infty / \beta_0^4 p_\infty) \quad \text{for } i_w \approx 0$$

$$R_\infty \gg 2M_\infty^2 / \beta_0^4 \text{ or } x \gg 1.5\,(U_\infty \mu_\infty / \beta_0^4 p_\infty) \quad \text{for } i_w = i_e \approx 0.17\, M_\infty^2 i_\infty$$

Such a procedure gives, generally speaking, the possibility of determining that distance from the nose $x = x_0$ at which one can then make use of the approximate equations of the method of small perturbations, but it has the defect inherent to all boundary layer theories that it does not take into account the effect of downstream separation on the point $x = x_0$ of either the exact solution or the approximate solution obtained above.

The limiting value $\eta_\delta = 0.98 \div 0.999$, assumed above, is to a certain extent conditional and is chosen from the following considerations.

For $M^2 = 1/a_2\beta_0^2 \gg 1$ the difference $i_\delta - 1 \quad (\sigma - 1)(1 - \eta_\delta)/a_2\beta_0^2$ is of order unity for $\eta_\delta = 0.98 \div 0.99$ and is close to zero for $\eta_\delta > 0.999$. Consequently, for $\eta \leqslant 0.98$ the effect of viscosity is quite noticeable, and for $\eta > 0.999$ it is negligibly small. On the other hand, because in the main body of the boundary layer $i - 1 \gg 1$, the boundary conditions can be considered to be satisfied in the indicated range both for the velocity and for the enthalpy.

We note that the magnitude of y_δ is practically unchanged in the limits $\eta_\delta = 0.98 \div 0.999$, which follows from a comparison of the functions h and h_0.

The method described may be applied also to blunt cones, if the degree of the bluntness is not great. In this case at some distance from the nose the field of flow, constructed without taking the boundary layer into account, will differ slightly from the conical flow field and the effect of this difference on the boundary layer can be taken into account independently in the linear construction [10].

Appendix to Section 3. We will examine the equation

$$g_*^2 j_m'' + (1 - \sigma) g_* g_*' j_m' + \beta_m \sigma\eta F(\eta) j_m = 1$$

or in self-conjugate form

$$(g_*^{1-\sigma} j_m')' + \sigma\beta m \frac{\eta F}{g_*^{\sigma+1}} j_m = 0 \tag{1}$$

This equation has two linearly independent solutions j_{1m} and j_{2m}, such that

$$j_{1m} \sim (-g_*')^{-\beta m}, \quad j_m \sim g_*^\sigma (-g_*')^{\beta m - 1} \quad \text{as } \eta \to 1$$

We will prove that the solution which satisfies the condition $j_m'(0) = 0$ can belong only to the type $j_{1m} = j_m{}^*$ if $0 < \beta_m < 1$.

Equation (1) reduces to the form:

$$\frac{d^2 j_m}{dt^2} = -\frac{\sigma\beta_m \eta F}{g_*^2} j_m, \quad t = \int_0^\eta g_*^{\sigma-1} d\eta$$

For $\eta < 1$ a theorem of Chaplygin is applicable to this equation, according to which $j_1 \leqslant j_2$ if $\beta_1 > \beta_2$ and $j_1 = j_2$, $j_1' = j_2'$ for $\eta = 0$.

For $\beta_m = 1$ equation (1) has the solution $j_0 = g_*^\sigma$. Using (1) we construct the difference:

$$(g_*^{1-\sigma} j_m')' j_0 - (g_*^{1-\sigma} j_0')' j_m = [g_*^{1-\sigma}(j_m' j_0 - j_0' j_m)]' (1 - \beta_m)\, \sigma \frac{\eta F}{g_*^{\sigma+1}} j_0 j_m \tag{2}$$

Integrating (3) for $j_m'(0) = 0$, we obtain

$$g_*^{(1-\sigma)}(j_m' j_0 - j_0' j_m) = (1-\beta_m)\,\sigma \int_0^\eta \frac{\eta F}{g_*^{\sigma+1}}\, j_0,\ j_m\, d\eta \tag{3}$$

If $j_m = j_{2m}$, then, as is easily seen, the left-hand side of (3) vanishes as $\eta \to 1$ like $g_*(-g_*')^{\beta_m - 1}$, and the integral on the right converges; therefore, the equality (3) is not possible. Consequently, $j_m = j_{1m}$; in this case the right-hand and left-hand sides of (3) are equal to $\sigma(-g_*')^{\beta_m - 1}$ for $\eta \approx 1$. Hence the assertion of Section 3 is proved. The functions j_m^* for $\sigma = 0.725$ are presented in Table 2:

TABLE 2.

η	0	0.20	0.40	0.60	0.70	0.80	0.90	0.95	0.97	0.99	1
j_1^*	1	0.996	0.980	0.943	0.910	0.864	0.790	0.725	0.684	0.616	0
j_2^*	1	0.999	0.992	0.959	0.927	0.884	0.817	0.762	0.715	0.648	0

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